JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 51, 68–75 (1975)

Existence of Tchebycheff Extensions

R. A. ZALIK

Department of Mathematics, University of the Negev, Beersheba, Israel

Submitted by Samuel Karlin

Given a Tchebycheff System $\{y_0 \cdots y_n\}$ defined on an interval I, it is proved that there exists a function y_{n+1}, such that the system $\{y_0 \cdots y_n, y_{n+1}\}$ is a Tchebycheff System on I. A function such as y_{n+1} is called a Tchebycheff extension of the system $\{y_i\}_{i=0}^n$.

1. INTRODUCTION

We start by recalling briefly some definitions and basic results which will be used in the sequel. For a more detailed discussion of the results quoted here, the reader is referred to the monograph by Karlin and Studden (1), in particular to Chapters I and XI.

A system of linearly independent continuous functions $\{y_i\}_{i=0}^n$ defined on an interval I (closed, open, or semiclosed) is called a Tchebycheff system, or a T system, provided that the determinants:

$$U\begin{pmatrix} y_0 & \cdots & y_n \\ t_0 & \cdots & t_n \end{pmatrix} = \begin{vmatrix} y_0(t_0) & \cdots & y_0(t_n) \\ y_n(t_0) & \cdots & y_n(t_n) \end{vmatrix} \tag{1.1}$$

maintain the same sign for every choice of points $t_0 < t_1 < \cdots < t_n$ of I. Without any loss of generality we may assume that they are strictly positive.

If the functions $\{y_i\}_{i=0}^n$ are of class $C^n(I)$, we can extend the definition of

$$U\begin{pmatrix} y_0 & \cdots & y_n \\ t_0 & \cdots & t_n \end{pmatrix}$$

as given by (1.1), so as to allow for equalities amongst the t_i. If $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ is any set of points of I, then

$$U^*\begin{pmatrix} y_0 & \cdots & y_n \\ t_0 & \cdots & t_n \end{pmatrix}$$

is defined to be the determinant in the right hand side of (1.1), where for each set of equal t_i's, the corresponding columns are replaced by the successive derivatives evaluated at the point. With this definition, the system $\{y_i\}_{i=0}^n$ will be called an Extended Tchebycheff system (ET system), provided that

$$U^*\begin{pmatrix} y_0 \cdots y_n \\ t_0 \cdots t_n \end{pmatrix} > 0,$$

for every set $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ of points of I. A system of functions $\{y_i\}_{i=0}^n$ such that for every $k = 0, 1,\ldots, n$, $\{y_i\}_{i=0}^k$ is a T system, will be called a complete Tchebycheff System (CT system). If, moreover, the systems $\{y_i\}_{i=0}^k$, $k = 0,\ldots, n$ are ET systems, then $\{y_i\}_{i=0}^n$ will be called an Extended complete Tchebycheff System (ECT system).

It can be proved that $\{y_i\}_{i=0}^n$ is an ECT system on the finite interval $[a, b]$, satisfying the initial conditions

$$y_i^{(p)}(a) = 0; \qquad p = 0, 1,\ldots, i-1; \quad i = 1, 2,\ldots, n, \tag{1.2}$$

if, and only if, the system $\{y_i\}_{i=0}^n$ has a representation in $[a, b]$ of the form

$$\begin{aligned} y_0(t) &= w_0(t), \\ y_1(t) &= w_0(t)\int_a^t w_1(s_1)\,ds_1, \\ &\vdots \\ y_n(t) &= w_0(t)\int_a^t w_1(s_1)\int_a^{s_1} w_2(s_2)\cdots\int_a^{s_{n-1}} w_n(s_n)\,ds_n\cdots ds_1, \end{aligned} \tag{1.3}$$

where the functions w_i can be expressed as the ratio of certain wronskians, namely

$$w_0 = y_0, \qquad w_1 = \frac{W(y_0, y_1)}{y_0^2},$$

$$w_i = \frac{W(y_0\cdots y_i)\,W(y_0\cdots y_{i-2})}{[W(y_0\cdots y_{i-1})]^2}, \qquad i = 2, 3,\ldots, n,$$

whence it is clear that they are strictly positive and continuous on $[a, b]$. We define the differential operators $D_i f = (d/dt)(1/w_i) f$, and note that, if $\{y_i\}_{i=0}^n$ has a representation of the form (1.3), then

$$D_0 y_0 = 0, \qquad D_0 y_1 = w_1,$$

and

$$D_0 y_i = w_1(t)\int_a^t w_2(s_1)\cdots\int_a^{s_{i-2}} w_i(s_{i-1})\,ds_1.$$

It is thus clear that also $\{D_0 y_i\}_{i=1}^n$ is an ECT system. The span of the system $\{y_i\}_{i=0}^n$ will be denoted by $L\{y_i\}_{i=0}^n$. If the conditions (1.2) are not satisfied by

$\{y_i\}_{i=0}^n$, and $\{y_i\}_{i=0}^n$ comprises an ECT system, we can subtract from each function y_i a suitable linear combination of the earlier functions to obtain an ECT system:

$$\tilde{y}_0 = y_0, \qquad \tilde{y}_i = y_i - \sum_{j=0}^{i-1} d_{ij} y_j, \qquad i = 1, 2, \dots, n,$$

which does satisfy (1.2). Clearly $\{\tilde{y}_i\}_{i=0}^n$ is a basis of $L\{y_i\}_{i=0}^n$.

A system $\{y_i\}_{i=0}^n$ is called a Weak Tchebycheff system (WT-system), provided that

$$U\begin{pmatrix} y_0 & \cdots & y_n \\ t_0 & \cdots & t_n \end{pmatrix} \geqslant 0$$

for every election $t_0 < t_1 < \cdots < t_n$ of points of I, and $y_i \in C[I]$, $i = 0, \dots, n$. A function y is said to be convex with respect to $\{y_i\}_{i=0}^n$, if $\{y_0, \dots, y_n, y\}$ is a WT system. The set of functions convex with respect to $\{y_i\}_{i=0}^n$ is evidently a cone. This cone is referred to as "generalized convexity cone," or "convexity cone," and is denoted by $C[y_0 \cdots y_n]$.

If f is any (real) function defined on the set I, by $\|f\|_I$ we shall understand the supremum of $|f(t)|$ taken over the set I. If there is no possibility of confusion, we shall simply write $\|f\|$.

2. Preliminary Lemmas

Lemma 1. *Let $\{y_i\}_{i=0}^n$ be a Tchebycheff System on an interval I. Assume that, for every set $t_0 < t_1 < \cdots < t_{n+1}$ of points of I, there is a bounded function x contained in the generalized convexity cone of $\{y_i\}_{i=0}^n$, such that*

$$U\begin{pmatrix} y_0, \dots, y_n, x \\ t_0, \dots, t_{n+1} \end{pmatrix} > 0.$$

Then $\{y_0, \dots, y_n\}$ has a Tchebycheff extension on I.

Proof. Let $\mathbf{t}$ denote any vector in R^{n+2}, and let P be the set of vectors $\mathbf{t} = (t_0, \dots, t_{n+1})$, such that all its coordinates belong to I, and $t_0 < t_1 < \cdots < t_{n+1}$. For every vector $\mathbf{t}$ of P, there exists a function $x_{\mathbf{t}}$ in the convexity cone of $\{y_0, \dots, y_n\}$, and a neighborhood $U_{\mathbf{t}} \in P$ of $\mathbf{t}$, such that, for any vector $\mathbf{t}' = (t_0', \dots, t_{n+1}')$ in $U_{\mathbf{t}}$,

$$U\begin{pmatrix} y_0, \dots, y_n, x_{\mathbf{t}} \\ t_0', \dots, t_n', t_{n+1}' \end{pmatrix} > 0.$$

The family $\{U_{\mathbf{t}}, \mathbf{t} \in P\}$ is a covering of P. Clearly, it has a denumerable subcovering $\{U_{\mathbf{t}_i}\}$, $i = 0, 1, 2, \dots$.

Setting

$$x(t) = \sum_{i=0}^{\infty} 2^{-i} \cdot \| x_{t_i} \|^{-1} \cdot x_{t_i}(t),$$

the conclusion follows. Q.E.D.

LEMMA 2. *Let $\{y_0, \ldots, y_n\}$ be a linearly independent set of real functions on an interval I, and assume that $\{u_0, \ldots, u_n\}$ is another basis of $L\{y_i\}_{i=0}^n$. Let A be the transition matrix, i.e.,*

$$\left\| \begin{matrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{matrix} \right\| = A \left\| \begin{matrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{matrix} \right\|.$$

If, for some set $\{t_0, \ldots, t_n\}$ of points of I,

$$\operatorname{sign} \bigcup \binom{u_0, \ldots, u_n}{t_0, \ldots, t_n} = \operatorname{sign} \bigcup \binom{y_0, \ldots, y_n}{t_0, \ldots, t_n} \tag{2.1}$$

and this sign is different from zero, then $\det A > 0$, *and therefore* (2.1) *holds for any set of points of I. Moreover, $\{y_0, \ldots, y_n\}$ is a (Weak) Tchebycheff System if, and only if, $\{u_0, \ldots, u_n\}$ is a (Weak) Tchebycheff System, and u belongs to the convexity cone of $\{y_0, \ldots, y_n\}$ if, and only if, it belongs to the convexity cone of $\{u_0, \ldots, u_n\}$. In addition to this, if $y_i \in C^{n-1}(I)$, then*

$$\operatorname{sign} W(y_0, \ldots, y_n)(t) = \operatorname{sign} W(u_0, \ldots, u_n)(t),$$

for every $t \in I$.

Proof. All the above assertions follow from the obvious equality

$$\bigcup \binom{u_0, \ldots, u_n}{t_0, \ldots, t_n} = (\det A) \cdot \bigcup \binom{y_0, \ldots, y_n}{t_0, \ldots, t_n}$$

valid for any set $\{t_0, \ldots, t_n\}$ of points of I. Q.E.D.

LEMMA 3. *Let $\{y_i\}_{i=0}^n$ be an* ECT *system on $[a, b]$, $n \geq 1$. Let*

$$V = \left(\sum_{i=0}^{n-1} \alpha_i y_i \right) + y_n$$

vanish at the points $b_0, b_1, \ldots, b_{n-1}$, where $a < b_0 < b_1 < \cdots < b_{n-1} < b$. Then, there exists a set $\{a_i\}_{i=1}^{n-1}$, $b_0 < a_{n-1} < \cdots < a_1 < b_{n-1}$, such that, for every $t \in [a, b]$,

$$V(t) = w_0(t) \int_{b_{n-1}}^{t} w_1(s_1) \int_{a_1}^{s_1} w_2(s_2) \cdots \int_{a_{n-1}}^{s_{n-1}} w_n(s_n)\, ds_n \cdots ds_1 .$$

Proof. Assume first that the functions y_i satisfy the initial conditions

$$y_i^{(p-1)}(a) = 0; \qquad i = 1,\ldots, p; \quad p = 1,\ldots, n.$$

Let $n = 1$. Then

$$V(t) = \alpha w_0(t) + w_0(t) \int_a^t w_1(s)\, ds,$$

whence $D_0 V = w_1$. Since $V(b_0) = 0$, it is clear that

$$V(t) = w_0(t) \int_{b_0}^t w_1(s)\, ds.$$

Assume now the assertion to be true for $n - 1$. Clearly $D_0 V$ vanishes at $n - 1$ points in the open interval (b_0, b_{n-1}). Hence, by inductive hypothesis,

$$D_0 V(t) = w_1(t) \int_{a_1}^t w_2(s_1) \cdots \int_{a_{n-1}}^{s_{n-2}} w_n(s_{n-1})\, ds_{n-1} \cdots ds_1,$$

where $b_0 < a_{n-1} < \cdots < a_1 < b_{n-1}$, and the assertion follows as before. In the general case, there is a basis $\{\tilde{y}_i\}_{i=0}^n$ of $L\{y_i\}_{i=0}^n$ of the form

$$\tilde{y}_0 = y_0, \qquad \tilde{y}_i = y_i - \sum_{j=0}^{i-1} \alpha_{ij} y_j, \qquad i = 1,\ldots, n,$$

satisfying the requested initial conditions. Trivially, V has a representation of the form $(\sum_{i=0}^{n-1} \beta_i \tilde{y}_i) + \tilde{y}_n$, whence the conclusion follows. Q.E.D.

DEFINITION 1.

$$y^{(k)}(t) = \int_a^b y(s)\, G_k(t - s)\, ds,$$

where

$$G_k(s) = (k/\sqrt{2\pi})\, e^{-\frac{1}{2}k^2 s^2}$$

is the Gauss Kernel.

DEFINITION 2. A function y_{n+1} will be called a Tchebycheff Extension of the system $\{y_0,\ldots, y_n\}$ on the interval I, if $\{y_0,\ldots, y_n, y_{n+1}\}$ is a T system on I.

3. EXISTENCE OF A TCHEBYCHEFF EXTENSION

THEOREM 1. *If $\{y_i\}_{i=0}^n$ is a T system on an interval I, it has a Tchebycheff Extension thereon.*

Proof. If $n = 0$, the assertion is trivial. Assume then that $n \geqslant 1$. Making if necessary an arctg change of variable, we can assume that I is bounded; a and b will denote its left and right end points. Without any loss of generality we can assume that $\{y_i\}_{i=0}^n$ is a Complete Tchebycheff System on (a, b); this follows from a theorem of Krein, (cf. [4, Corollary on p. 310]), and an obvious application of Lemma 2. Dividing if necessary each function by $|y_0| + |y_1| + \cdots + |y_n|$, we can assume the functions y_i to be bounded on I.

From the Generalized Composition Formula (cf. [1, p. 14]), it is clear that $\{y_i^{(k)}\}_{i=0}^n$ is an Extended Complete Tchebycheff System on $[a, b]$, for $k = 1, 2,\ldots$.

Let a set $t_1 < t_2 < \cdots < t_n$ of points of I be given, and let

$$u^{(k)}(t) = \bigcup\begin{pmatrix} y_0^{(k)},\ldots, y_n^{(k)} \\ t_1 ,\ldots, t_n , t\end{pmatrix}\Big/\bigcup\begin{pmatrix} y_0^{(k)},\ldots, y_{n-1}^{(k)} \\ t_1 ,\ldots, t_n\end{pmatrix}. \tag{3.1}$$

It is obvious that the sequence $\{\| u^{(k)} \|_{[a,b]}\}_{k=1}^\infty$ is bounded. From (3.1) and Lemma 3, we conclude that

$$u^{(k)}(t) = w_0^{(k)}(t) \int_{t_n}^t w_1^{(k)}(s_1) \int_{a_1^{(k)}}^{s_1} w_2^{(k)}(s_2) \cdots \int_{a_{n-1}^{(k)}}^{s_{n-1}} w_n^{(k)}(s_n)\, ds_n \cdots ds_1 ,$$

where

$$t_1 < a_{n-1}^{(k)} < \cdots < a_1^{(k)} < t_n .$$

Clearly, if $t \geqslant t_n$,

$$\begin{aligned} u^{(k)}(t) &\geqslant w_0^{(k)}(t) \int_{t_n}^t w_1^{(k)}(s_1) \int_{t_n}^{s_1} w_2^{(k)}(s_2) \cdots \int_{t_n}^{s_{n-1}} w_n^{(k)}(s_n)\, ds_n \cdots ds_1 \\ &= V^{(k)}(t) \geqslant 0. \end{aligned}$$

It is therefore clear that the sequence $\{\| V^{(k)} \|_{[t_n,b]}\}_{k=1}^\infty$ is bounded. However, $V^{(k)}$ has a representation of the form

$$V^{(k)} = \sum_{i=0}^n a_i^{(k)} y_i^{(k)}$$

with $a_n^{(k)} = 1$ (cf. [1, Chapter XI, Remark 1.2]) whence we can easily show that the coefficients $a_i^{(k)}$ are uniformly bounded on i and k. It is therefore clear that there exist numbers $a_1 ,\ldots, a_n$, with $a_n = 1$, and a subsequence k_r, such that

$$\lim_{r\to\infty} V^{(k_r)} = \sum_{i=0}^n a_i y_i \qquad \text{on} \qquad (a, b).$$

Now, let $\varphi^{(k)}(t_n; t)$ and $V(t)$ be defined by:

$$\varphi^{(k)}(t_n; t) = \begin{cases} 0, & a \leqslant t < t_n, \\ V^{(k)}(t), & t_n \leqslant t \leqslant b, \end{cases}$$

and

$$V(t) = \begin{cases} 0, & t \in [a, t_n], \\ \sum_{i=0}^{n} a_i y_i(t), & t \in [t_n, b] \cap I. \end{cases}$$

Note that V is bounded on I.

Since $V^{(k)}(t_n) = 0$, it is clear that V is continuous on $[a, b]$, except perhaps at b, if it isn't contained in I. Clearly,

$$\lim_{r\to\infty} \varphi^{(k_r)}(t_n; \cdot) = V \qquad \text{on} \qquad [a, b).$$

It is well known that

$$\varphi^{(k_r)}(t_n; \cdot) \in C[y_0^{(k_r)}, \ldots, y_n^{(k_r)}],$$

as follows readily by Lemma 2 and [1, Ch. XI, Lemma 2.1 and Remark 1.2.] Therefore $V \in C[y_0, \ldots, y_n]$.

We shall now show that V is strictly positive on $(t_n, b] \cap I$. Assume that $V(t') = 0$, for some point $t' \in (t_n, b]$. By the definition of V, it is clear that it has at most n zeros on the interval (t_n, b), thus it is clear that there exists a point $t'' \in (t_n, t')$, such that $V(t'') \neq 0$. Since $V^{(k_r)}$ is positive on $(t_n, b]$, passing to the limit we conclude that $V(t'') > 0$. Let $a \leqslant s_0 < \cdots < s_{n-1} \leqslant t_n$. Since V vanishes on $[a, t_n]$ and is contained in $C[y_0, \ldots, y_n]$ on I, we have:

$$0 \leqslant U\begin{pmatrix} y_0, \ldots, y_n, V \\ s_0, \ldots, s_{n-1}, t'', t' \end{pmatrix} = -V(t') \cdot U\begin{pmatrix} y_0, \ldots, y_n \\ s_0, \ldots, s_{n-1}, t' \end{pmatrix} < 0,$$

which is a contradiction. Thus V is strictly positive on $(t_n, b]$, and in particular at t_{n+1}. Hence, since $V(t_i) = 0$, $i = 0, \ldots, n$,

$$U\begin{pmatrix} y_0, \ldots, y_n, V \\ t_0, \ldots, t_{n+1} \end{pmatrix} = V(t_{n+1}) \cdot U\begin{pmatrix} y_0, \ldots, y_n \\ t_0, \ldots, t_n \end{pmatrix} > 0.$$

Since the set $\{t_i\}_{i=0}^{n+1}$ is arbitrary, the conclusion follows from Lemma 1.

Q.E.D.

Acknowledgments

This work is based on techniques developed by the author in his doctoral thesis, which was submitted to the Senate of the Technion-Israel Institute of Technology; in particular, Lemmas 1 and 2, as well as a special case of Theorem 1 can be found therein.

The author takes advantage of this opportunity to express his heartfelt gratitude to his adviser, Professor Zvi Ziegler of the Faculty of Mathematics of the Technion for his guidance and encouragement, and to thank Dr. Alan Solomon, of the Department of Mathematics of the University of the Negev, who suggested an improvement.

Note added in proof. While this work was being processed for publication, there appeared a paper by Ronald Zielke in which a different proof of Thm. 1 is given (see R. Zielke, Alternation Properties of Tchebyshev-Systems and the Existence of Adjoined Functions, *J. Approximation Theory* **10** (1974), 172–184).

References

1. S. Karlin and W. Studden, "Tchebycheff Systems, with Applications in Analysis and Statistics," Interscience, New York, 1966.
2. R. C. Jones and L. A. Karlovitz, Equioscillation under nonuniqueness in the approximation of continuous functions, *J. Approximation Theory* **3** (1970), 138–145.
3. K. P. Hadeler, Remarks on Haar systems, *J. Approximation Theory* **7** (1973), 59–62.
4. A. B. Németh, About the extension of the domain of definition of the Chebyshev systems defined on intervals of the real axis, *Mathematica* **11** (1966), 307–310.